Stochastic dynamics of barrier island elevation

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Barrier islands are ubiquitous coastal features that create low-energy environments where salt marshes, oyster reefs, and mangroves can develop and survive external stresses. Barrier systems also protect interior coastal communities from storm surges and wave-driven erosion. These functions depend on the existence of a slowly migrating, vertically stable barrier, a condition tied to the frequency of storm-driven overwashes and thus barrier elevation during the storm impact. The balance between erosional and accretional processes behind barrier dynamics is stochastic in nature and cannot be properly understood with traditional continuous models. Here we develop a master equation describing the stochastic dynamics of the probability density function (PDF) of barrier elevation at a point. The dynamics are controlled by two dimensionless numbers relating the average intensity and frequency of high-water events (HWEs) to the maximum dune height and dune formation time, which are in turn a function of the rate of sea level rise, sand availability, and stress of the plant ecosystem anchoring dune formation. Depending on the control parameters, the transient solution converges toward a high-elevation barrier, a low-elevation barrier, or a mixed, bimodal, state. We find the average after-storm recovery time—a relaxation time characterizing barrier's resiliency to storm impacts—changes rapidly with the control parameters, suggesting a tipping point in barrier response to external drivers. We finally derive explicit expressions for the overwash probability and average overwash frequency and transport rate characterizing the landward migration of barriers.

Barrier islands | stochastic dynamics | coastal dunes | master equation

Barrier island elevation is determined by the competition between the formation of vegetated dunes or foredunes (the highest natural feature on a barrier) and random water-driven erosional events (Fig. 1*A*). Vegetated dunes form when plants trap wind-blown sand and their growth thus depends on the establishment of a dune-building plant ecosystem, the availability of fine sand, and the presence of a dry beach (1). Fast-growing dunes can recover before the next storm or high-water event (HWE) hits the island, in which case islands will tend to have well-developed dunes, resist storm impacts, migrate slowly (if at all), and support a rich ecosystem and/or human development. In contrast, slow-growing dunes can be frequently eroded, which keeps island elevation low and prone to frequent overwash, resulting in rapid landward migration and low biodiversity. These two extreme cases can be associated with high-elevation and low-elevation barrier states, respectively (2) (Fig. 1*A*).

HWEs—defined by clusters of total water levels above a given threshold elevation—can be divided into two broad groups based on the relation between maximum total water level, beach elevation, and the height of mature dunes (3). The first group consists of interannual high-intensity events (e.g., large storms) overtopping and potentially eroding mature dunes. The second group consists of intraannual low-intensity events flooding the beach, which can disrupt after-storm dune recovery when barrier elevation is low (Fig. 1*A*).

Recent measurements of the stochastic properties of intraannual HWEs in several locations around the globe, reported in a companion study in PNAS (3), show they can be modeled as a marked Poisson process with exponentially distributed marks. The mark of a HWE is defined as the maximum water level above the beach during the duration of the event and characterizes its size and intensity. This result opens the way for a probabilistic model of the temporal evolution of the barrier/dune elevation, along the lines of the stochastic model of soil moisture dynamics (4). The knowledge of the transient probability distribution function of barrier elevation allows the calculation of after-storm recovery times, overwash probability and frequency, and the average overwash transport rate driving the landward migration of barriers.

Master Equation for Barrier Elevation

In general, the barrier elevation z at a point (x, y) is a function of the cross-shore x and along-shore y position (Fig. 1*B*) and its temporal evolution has to be calculated with complex eco-morphodynamic models (1, 2). In what follows, we propose several approximations to reduce this complex two-dimensional problem to a point (zero-dimensional) description.

Following ref. 1, we assume dunes form at a given cross-shore location $x_c(y)$, dictated by where vegetation can survive long enough to form a dune. Since this location is also the position of the dune crest (1), the highest point of a barrier island at a given along-shore position y can be represented by the dune height $h(y, t) = z(x_c, y, t)$ (Fig. 1*B*). For simplicity, we consider dune elevation relative to a washover fan (i.e., a bare low-elevation area), such that the condition $h = 0$ represents the absence of a dune and thus the outcome of an overwash.

Numerical simulations (1) and field measurements (5) show dune growth in the absence of wave overtopping has a characteristic time T_d and saturates at a maximum dune height H (for a steady shoreline). The dune growth rate (ρ) can thus be approximated by

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Significance

Barrier islands sustain important coastal ecosystems and are our first line of defense from storm impacts. However, they are threatened by sea level rise and lack of sand supply and could undergo rapid deterioration. The uncertainty in barrier response is amplified by the complexity of the processes involved and their stochastic nature. Here we find that barrier stochastic behavior can be described by a stochastic equation that suggests the presence of a tipping point in barrier response to external drivers. The solution of this equation allows the quantitative estimation of after-storm recovery, the rate of barrier landward migration, the effectiveness of dune protection in reducing back-barrier flooding, and the effects of potential interventions to accelerate after-storm recovery.

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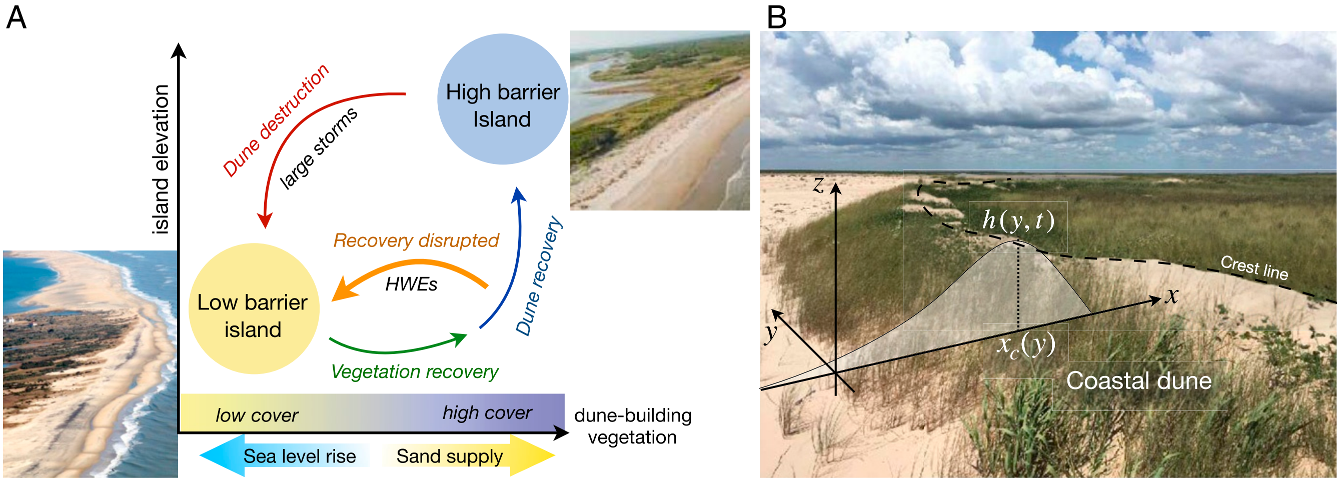

Fig. 1. (*A*) Simplified dynamics of barrier elevation. Extreme HWEs, i.e., large storms, erode the mature dunes that define a high-elevation barrier, along with the dune-building vegetation; whereas frequent low-intensity HWEs disrupt after-storm dune recovery and keep the island in a low-elevation bare state. Sand supply increases dune growth and promotes a high-elevation state. Sea level rise promotes a low barrier by increasing plant stress and reducing the availability of dry sand. Examples of high and low barriers are from Virginia. (*B*) Geometrical description of coastal dunes where x_c is the cross-shore position of dune crest h. Image is from Cedar Lakes, TX.

$$\mathrm{d}h/\mathrm{d}t \equiv \rho(h) = (H-h)/T_d. \quad [1]$$

The undisturbed dune growth after an overwash ($h(0)=0$) thus follows the curve $h(t)=H\left(1-e^{-t/Td}\right)$. In a first approximation we neglect the complexities of vegetation dynamics during the recolonization of the washover fan (2, 5) and consider the initial plant establishment is much faster than the ensuing dune growth. The implications of this assumption are explored in *Discussion and Conclusions*.

The along-shore profile of barrier elevation is assumed to consist of different random realizations of the stochastic dune elevation $h(t)$ at a point. These dynamics are the result of dune-forming processes, characterized by the continuous wind-driven deposition rate $\rho(h)$ that increases dune size by $\mathrm{d}h$ in a time interval $\mathrm{d}t$ (Eq. **1**), and random erosional HWEs decreasing dune size by an amount $\Delta h>0$ when an event i occurs at time $t=t_i$,

$$\mathrm{d}h = \rho(h)\mathrm{d}t - \Delta h(h,t). \quad [2]$$

We model HWEs as a marked Poisson process with frequency λ_0 and exponentially distributed sizes ("marks") S above the beach (3). Consistent with our point description, we neglect horizontal dune erosion—which could be relevant in determining the outcome during long or closely spaced storms—and assume dune height remains unchanged if no significant overtopping occurs ($S<rh$), whereas the dune is completely eroded otherwise ($S>rh$), with overwash parameter r between 1 and 2 (6). This is consistent with semiempirical models of dune erosion (2) and represents a sensible idealization of both field data (6) and conceptual storm impact scales for barrier islands (7). For simplicity in the nomenclature, in what follows we refer to any significant overtopping event ($S>rh$) as an overwash event, even for negligible dune height ($h\sim 0$) where the term flooding could be more appropriate.

The stochastic erosional dynamics can be included into a Chapman–Kolmogorov forward equation for the evolution of the probability density function (PDF) of barrier elevation $f(h,t)$ (*Materials and Methods*). After rescaling dune height by H and time by T_d, the master equation for $h>0$ reads

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \xi}[\rho^+(\xi)f(\xi,t)] - \lambda^+(\xi)f(\xi,t), \quad [3]$$

where $\xi=h/H$ is the rescaled dune height, $\rho^+(\xi)=1-\xi$ is the rescaled dune growth rate (Eq. **1**), and the term $\lambda^+(\xi)=\lambda_0^+e^{-\xi/\xi_c}$, with $\xi_c=\bar{S}/(rH)$ and $\lambda_0^+=\lambda_0 T_d$, represents the rescaled frequency of events eroding a dune of rescaled height ξ. For simplicity, we keep the rescaled time as t.

Integrating Eq. **3** over ξ and using the normalization condition $\int_0^1 f(\xi,t)\mathrm{d}\xi=1$, we get the boundary condition at $\xi=0$,

$$\rho^+(0)f(0,t) = \int_0^1 \lambda^+(\xi)f(\xi,t)\mathrm{d}\xi \equiv \overline{\lambda^+}(t), \quad [4]$$

which states that the probability density of having no dunes $f(0,t)$ equals the ratio of the expected overwash frequency $\overline{\lambda^+}$ and the maximum dune growth rate $\rho^+(0)=1$.

Control Parameters. The evolution of the probability density function $f(\xi,t)$ is completely characterized by two dimensionless parameters: the rescaled mean size (intensity) of HWEs, $\xi_c=\bar{S}/(rH)$, also representing the rescaled height of a dune eroded by the average HWE ($\bar{S}$), and the rescaled frequency $\lambda_0^+=\lambda_0 T_d$ representing the ratio between the characteristic dune formation time T_d and the average time $1/\lambda_0$ between HWEs. Global measurements of HWEs (3) suggest both $\bar{S}$ and λ_0 (in the range 0.2 to 0.4 m and 1 to 2 mo^{-1}, respectively) change little with wave and tidal conditions. Therefore, the control parameters $\xi_c \propto H^{-1}$ and $\lambda_0^+ \propto T_d$ can be interpreted as proxies for dune size (H) and formation time (T_d). Note that dune size depends on plant zonation and thus on the stress of the plant ecosystem anchoring dune formation (1), whereas dune formation time depends on the wind regime and the availability of dry sand (1). Both quantities can thus be affected by sea level rise as the water table becomes shallower and the stress on salt-intolerant vegetation increases.

Solution, Interpretation, and Derived Quantities

Transient Solution. Eq. **3** is solved using the method of characteristics, where the curve $\xi^*(t)=1-e^{-t}$ divides the (ξ,t) plane into two regions mapping either to the initial condition $f(\xi,0)$ for $\xi \geq \xi^*(t)$ or to the boundary condition $f(0,t)$ for $\xi<\xi^*(t)$. Here

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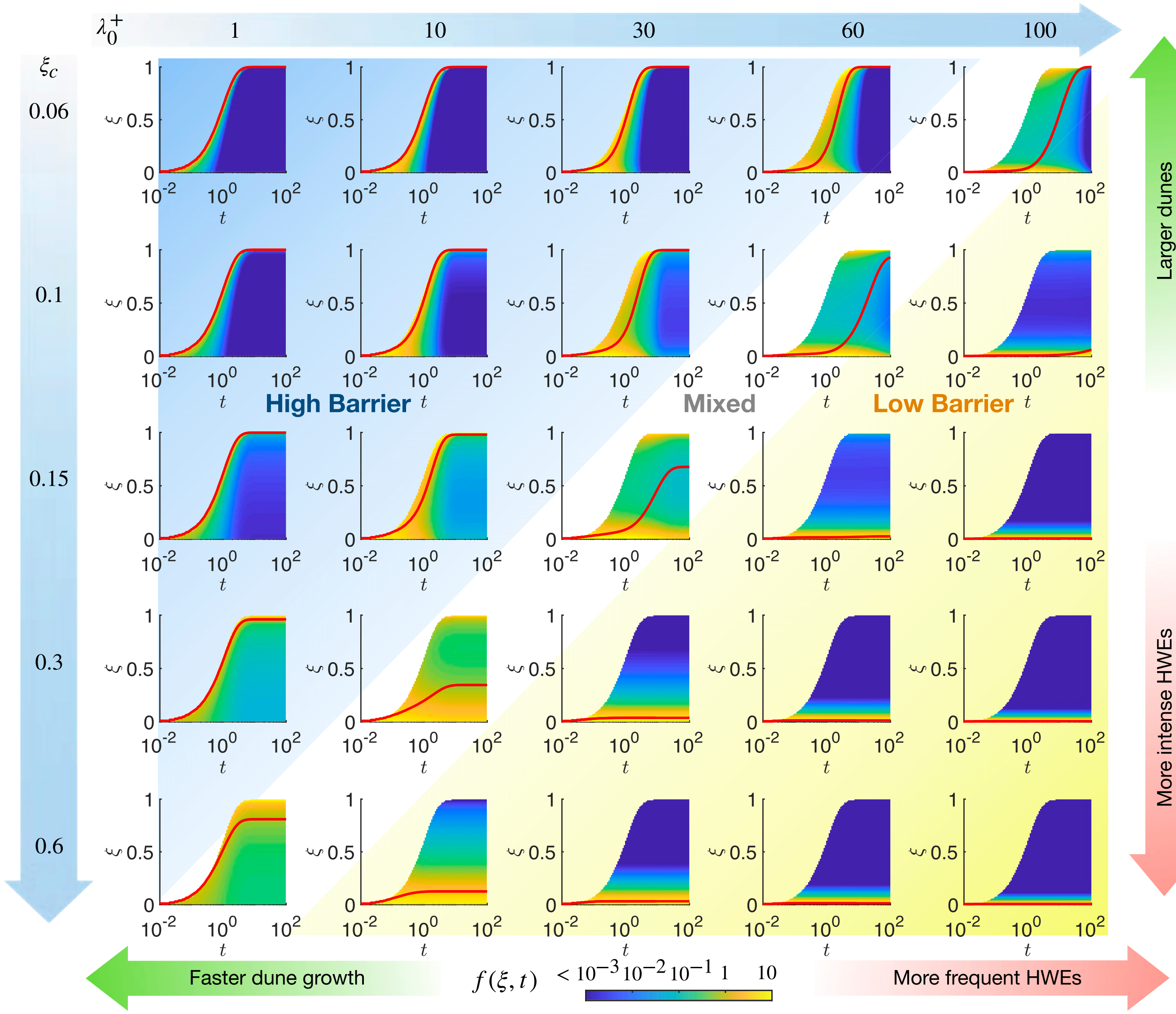

Fig. 2. Color plots of the temporal evolution of the PDF ($f(\xi, t)$) of rescaled barrier elevation ξ after an overwash (Eqs. **7** and **8**) for different rescaled HWE frequencies $\lambda_0^+ = \lambda_0 T_d$ (columns) and rescaled mean HWE intensities $\xi_c = \bar{S}/(rH)$ (rows) (values shown on top and left, respectively). Solid lines show the expected values $\bar{\xi}(t)$. The regions of the parameter space corresponding to a high, low, and mixed (bimodal PDF) barrier are highlighted. Arrows on bottom and right show the effects of changing maximum dune size (H), dune growth time (T_d), and frequency (λ_0) and intensity (i.e., mean size $\bar{S}$) of HWEs.

we focus on initial conditions of the form $f(\xi, 0) = \delta(\xi - \xi_0)$, representing the outcome of an overwash ($\xi_0 = 0$; Fig. 2) or dune restoration project ($\xi_0 > 0$).

The transient PDF for $\xi \geq \xi^*(t)$ is the probability density of the initial elevation $\xi = \xi_0$ not being eroded while following the deterministic growth curve $\xi(t) = 1 - (1 - \xi_0)e^{-t}$:

$$f(\xi, t) = p(t, \xi_0)\, \delta(\xi - \xi(t)), \qquad [5]$$

where $p(t, \xi_0) = e^{-\int_0^t \lambda^+(\xi(t))\mathrm{d}t}$ represents the probability of not having an overwash before time t. Using the definition $\mathrm{d}\xi = \rho^+(\xi)\mathrm{d}t$, this probability can be written as

$$p(t, \xi_0) = \phi(\xi(t))/\phi(\xi_0), \qquad [6]$$

with the auxiliary function $\phi(\xi)$ defined in *Materials and Methods*.

For $\xi < \xi^*(t)$, the transient PDF depends on the values $f(0, t)$ at the boundary $\xi = 0$ and has the form

$$f(\xi, t) = f(0, t - t'(\xi))\, \phi(\xi)/\rho^+(\xi), \qquad [7]$$

where $t'(\xi) = -\ln(1 - \xi)$. The values $f(0, t)$ are obtained from the normalization condition $\int_0^1 f(\xi, t)\mathrm{d}\xi = 1$, leading to the integral equation

$$\int_0^t \phi(\xi^*(t'))f(0, t - t')\mathrm{d}t' = 1 - p(t, \xi_0). \qquad [8]$$

Steady-State Solution. Taking $\partial f/\partial t = 0$ in Eq. **3**, we arrive at the steady-state solution f_∞:

$$f_\infty(\xi) = \left(\int_0^1 \frac{\phi(\xi)}{\rho^+(\xi)}\mathrm{d}\xi\right)^{-1} \frac{\phi(\xi)}{\rho^+(\xi)}. \qquad [9]$$

This solution has a minimum at $\xi_{\min} = \xi_c \ln \lambda_0^+$ and therefore is strictly bimodal for $0 < \xi_{\min} < 1$. For $\xi_{\min} \leq 0$ (i.e., $\lambda_0^+ \leq 1$) f_∞ has a single high-elevation mode (at $\xi = 1$), whereas for $\xi_{\min} \geq 1$ (i.e., $\lambda_0^+ \geq e^{1/\xi_c}$) there is a single low-elevation mode (at $\xi = 0$).

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Intermodal Transition Times and Barrier Elevation Regimes. In the parameter subset where the steady-state PDF exhibits bimodality ($0<\xi_{\min}<1$), the mean excursion time $\overline{T}_l$ below the minimum of the steady state ($\xi_{\min}$) provides an indication of the timescale associated with bimodal switching, from the low- to the high-elevation mode.

Following ref. 8, $\overline{T}_l$ is calculated as the ratio of the fraction of time the system spends below $\xi_{\min}$, given by the cumulative distribution $P_\infty(\xi_{\min})=\int_0^{\xi_{\min}} f_\infty(\xi')\mathrm{d}\xi'$, and the mean rate $\nu(\xi_{\min})$ of up-crossings of the level $\xi_{\min}$, $\nu(\xi_{\min})=\rho^+(\xi_{\min})f_\infty(\xi_{\min})$. Substituting Eq. **9** into the definition of $\overline{T}_l$ gives

$$\overline{T}_l=\frac{1}{\phi(\xi_{\min})}\int_0^{\xi_{\min}}\frac{\phi(\xi')}{\rho^+(\xi')}\mathrm{d}\xi'. \qquad \textbf{[10]}$$

Outside the bimodal region, $\overline{T}_l\equiv 0$ when f_∞ has a single high-elevation mode ($\xi_{\min}\le 0$ for $\lambda_0^+\le 1$), and $\overline{T}_l\to\infty$ when f_∞ has a single low-elevation mode ($\xi_{\min}\ge 1$ for $\lambda_0^+\ge e^{1/\xi_c}$).

The behavior of $\overline{T}_l$ suggests three distinct regimes for the dynamics of barrier elevation, what we call a "high," a "low," and a "mixed" barrier (Figs. 2 and 3). In the cases where $\overline{T}_l\gg 1$, the process of modal switching is long relative to the explicit timescales of the dynamics, namely $T_D=1$ (by definition of the rescaling) and $1/\lambda_0^+<1$, and the barrier remains low elevation and prone to overwashes ("low barrier"). For $\overline{T}_l\ll 1$, the opposite is true; modal switching is fast and the barrier tends to be high elevation most of the time ("high barrier"). Finally, for intermediate timescales ($\overline{T}_l\sim 1$ to 10) both modes are relevant and the barrier remains in a mixed (bimodal) state alternating between high and low elevations ("mixed barrier"; Fig. 2). Note that these regimes include the transient dynamics and do not necessarily correspond to the modal properties of the steady-state solution.

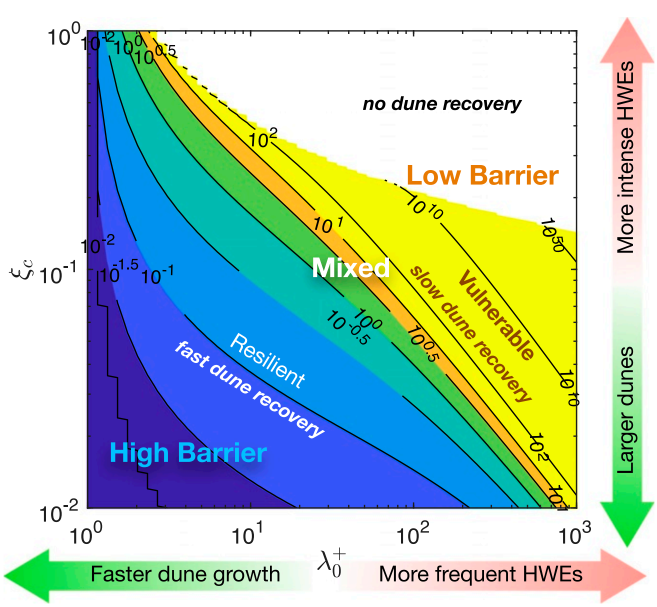

Fig. 3. Contour lines of the mean excursion time $\overline{T}_l$ within the low-elevation mode, also identified as the mean after-storm recovery time, rescaled by the dune formation time T_d, as a function of the two control parameters. Regions corresponding to resilient high barriers ($\overline{T}_l\lesssim 1$), vulnerable low barriers ($\overline{T}_l\gtrsim 10$), and mixed (bimodal) barriers are highlighted. The white region corresponds to $\lambda_0^+\ge e^{1/\xi_c}$ where $\overline{T}_l\to\infty$ and thus dunes never recover. Arrows show the effects of dune and HWEs characteristics on the control parameters (see Fig. 2 for more details).

After-Storm Recovery Time and Tipping Point in Barrier Response. The mean excursion time $\overline{T}_l$ within the low-elevation mode can also be interpreted as the mean after-storm dune (or barrier) recovery time. Therefore, the intermodal transition time provides a characterization of the resiliency of the barrier under a combination of external and intrinsic conditions described by the control parameters (Fig. 3). For $\overline{T}_l\lesssim 1$, barrier elevation recovers quickly after an overwash and can be classified as resilient, whereas for $\overline{T}_l\gtrsim 10$ recovery takes much longer as a now vulnerable barrier experiences frequent overwashes. The faster-than-exponential increase of the recovery time in this region (Fig. 3) can be interpreted as the crossing of a tipping point in barrier response.

The characteristics of the tipping point can be derived analytically for $\xi_c\ll 1$. In that case, the exponential integral $E_i(1/\xi_c)$ in Eq. **17** can be approximated by $\xi_c e^{(1/\xi_c)}$, leading to $\overline{T}_l\propto e^{\lambda_0^+\xi_c\beta(\xi_c)}$, where β is a weak function of ξ_c. In a first approximation, the tipping point in barrier response, roughly defined by the boundary $\overline{T}_l\sim 10$ limiting resilient barriers, is thus given by a constant value of the product of the two control parameters: $\xi_c\lambda_0^+$ (Fig. 3).

The emergence of strongly time-separated modes ($\overline{T}_l\gg 1$) above the tipping point has important implications for the system. It may inform approaches to remediation and management, in which circumstance it would be useful to know how high a dune needs to be to be likely to persist, but it also presents a quandary: The observation window in time necessary to resolve the predicted distribution may be impractically long. Rather, spatial statistics must serve as a proxy, as highlighted in *Discussion and Conclusions*.

Average Overwash Frequency and Transport Rate. In addition to describing the natural variability of barrier elevation, the transient PDF $f(\xi,t)$ also defines the expected overwash frequency at time t: $\overline{\lambda^+}(t)=f(0,t)$ (Eq. **4**). This allows the calculation of the average overwash frequency $\langle\lambda^+\rangle_T$ during a time interval T after an overwash as $\langle\lambda^+\rangle_T=\frac{1}{T}\int_0^T\overline{\lambda^+}(t)\mathrm{d}t=\frac{1}{T}\int_0^T f(0,t)\mathrm{d}t$, which provides information about the role the intermittent presence or absence of dunes can play in promoting long-term barrier habitat resilience.

The average overwash frequency also quantifies the exchange of sediments between the beach and the back barrier leading to the landward migration of barriers. Indeed, as shown in *Materials and Methods*, the expected value of the transport rate $\overline{Q}_S(t)$ per HWE can be approximated as $\overline{Q}_S(t)=Q_0\overline{\lambda^+}(t)/\lambda_0^+$ with scaling term Q_0. Thus, the average transport rate $\langle Q\rangle_T$ during a time interval T after an overwash is

$$\langle Q\rangle_T=Q_0\langle\lambda^+\rangle_T/\lambda_0^+ \qquad \textbf{[11]}$$

and the cumulative transport rate resulting from all HWEs λ_0^+T is simply $\langle Q\rangle_T\lambda_0^+T=Q_0\langle\lambda^+\rangle_T T$.

Overwash Probability during Dune Recovery and Effects of Potential Interventions. A final metric to study the vulnerability of dune recovery and the effectiveness of potential interventions is the probability $p_e(t,\xi_0)$ of an overwash taking place before a time t for a dune of initial elevation ξ_0:

$$p_e(t,\xi_0)=1-p(t,\xi_0)=1-\phi(\xi(t))/\phi(\xi_0), \qquad \textbf{[12]}$$

where p is the probability of no erosion (Eq. **6**) and $\xi(t)=1-(1-\xi_0)e^{-t}$ is the deterministic growth of the initial condition $\xi(0)=\xi_0$. Defining the dune recovery height as $\xi_r=1-e^{-1}$, the relevant probability to evaluate the vulnerability of a dune of size ξ_0 is the probability $p_{e_r}=p_e(t_r,\xi_0)$ of

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an overwash occurring before the dune recovers, where $t_r = 1 + \ln(1 - \xi_0)$ is the deterministic recovery time satisfying the condition $\xi(t_r) = \xi_r$.

As expected, p_{e_r} increases with both control parameters, λ_0^+ and ξ_c, in response to more intense and frequent HWEs and/or smaller and slowly growing dunes (Fig. 4*A*). Furthermore, p_{e_r} decreases with the size (ξ_0) of initial dunes (Fig. 4*B*). Interpreting initial dune size as the degree of intervention in the system, p_{e_r} thus provides a measure of its effectiveness to minimize dune overtopping and overwash.

For $\xi_c \ll 1$ and $\xi_0 \ll 1$, p_{e_r} can be approximated by an exponential saturation of the form

$$p_{e_r} \approx 1 - \exp\left(-\lambda_0^+ \xi_c e^{-\xi_0/\xi_c} \gamma(\xi_c, \xi_0)\right), \qquad [13]$$

where γ is a weak function of ξ_c and ξ_0, and we use the approximation $E_i(x \gg 1) \approx e^x/x$ in Eq. **17**. The relation $\lambda_0^+ \xi_c \propto e^{\xi_0/\xi_c}$ thus describes the lines of constant p_{e_r} in the parameter space (contour lines in Fig. 4) and provides a good approximation to constrain the regions of negligible or highly probable overwash.

Discussion and Conclusions

Our results complement those of a previous process-based model (2) focused on the controls of after-storm vegetation dynamics. Together, they propose two complementing mechanisms for the slowdown in dune growth (Fig. 1*A*): first, the slow recovery of vegetation (2), which is required for dune formation and was assumed to depend on wind-driven sand accretion in the absence of plants (2), and second, direct water-driven erosion of small vegetated dunes (proto-dune), which we find can lead to a low-elevation barrier even for fast vegetation dynamics and is the focus of the present study. These two studies thus explore different aspects of a more general parameter space involving biophysical interactions. However, by characterizing the conditions leading to a low-elevation mode during fast vegetation growth, we provide a sufficient condition for the existence of a low-elevation barrier. Indeed, a slower vegetation recovery will only increase the dominance of the low-elevation mode.

Furthermore, the stochastic dynamics of dune growth given here, and the probabilistic characterizations thence derived, relate to a single point, as stated at the outset. The observational quantity corresponding to this model would be a time series of heights for a dune that is (piecewise) homogeneous in the y direction. In particular, the predicted steady-state distribution should be compared with the empirical distribution function of such a time series with a suitably large number of sample points. This comparison would allow for parameter estimation or model validation. However, as was previously cautioned, a dune in the mixed regime presents a nontrivial distribution structure and potentially takes a very long time to explore its state space if the modes are well separated, so that the observational series needed to resolve it would be very long indeed. Somehow, the longitudinal distribution of dune heights must be made to serve instead. This will be the topic of a forthcoming paper.

In summary, here we developed and solved a master equation describing the temporal evolution of the PDF of barrier elevation, represented by the height of its coastal dune. We find, consistent with earlier work (2), that barriers can be low elevation, high elevation, or mixed (bimodal PDF), depending on two control parameters that characterize the competition between vertical accretional and erosional processes in terms of the ratio of the temporal and spatial scales of HWEs and coastal dunes. The transient PDF and derived quantities such as intermodal transition times or after-storm recovery times, average overwash frequency, and overwash probability before dune recovery provide a quantitative description of barrier state and its resilience/vulnerability. These quantities can be used to evaluate important and varied aspects of barrier response such as the rate of barrier migration due to overwash-related sand transport, the effectiveness of dune protection in reducing back-barrier flooding, and the effects of potential interventions to accelerate after-storm recovery. Furthermore, this model allows the study of coastal remediation strategies using the initial condition to parameterize single interventions or modifying the effects of erosional events to account for multiple interventions. Our analytical results open the door to simplified mean-field models of barrier and barrier systems including dune–beach interactions and the coupled barrier–marsh–lagoon system.

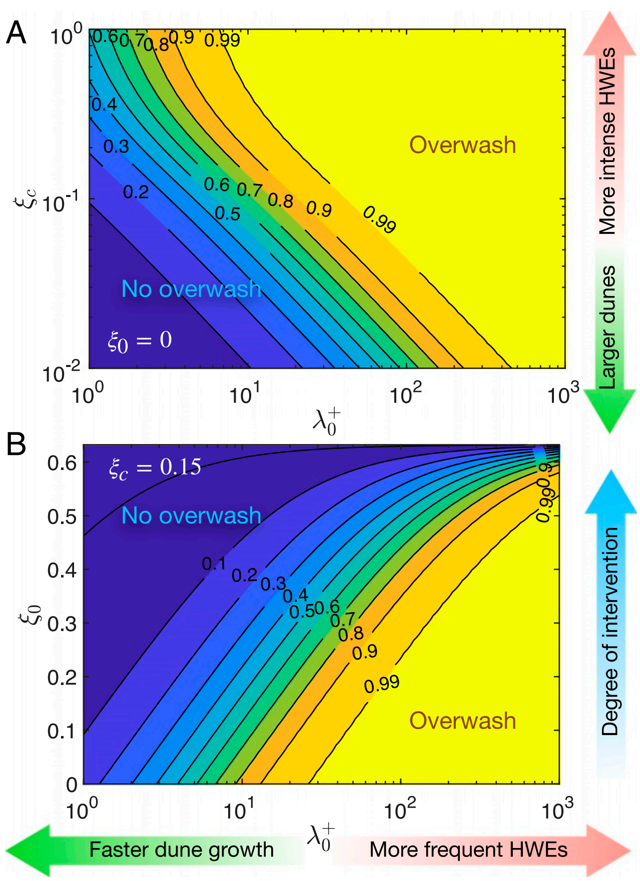

Fig. 4. (*A* and *B*) Contour lines of the probability p_{e_r} of an overwash taking place before dune recovery, as a function of the control parameters λ_0^+ and ξ_c for an initial dune size $\xi_0 = 0$ (*A*) and a function of λ_0^+ and the initial dune size ξ_0 for $\xi_c = 0.15$ (*B*). Arrows show the effects of dune and HWEs characteristics on the control parameters (see Fig. 2 for more details).

Materials and Methods

Derivation of the Master Equation. The probability density $f_E(\Delta h, h)$ of a dune h being either completely eroded ($\Delta h = h$) or not eroded at all ($\Delta h = 0$) after a HWE is given by

$$f_E(\Delta h, h) = \delta(\Delta h) \int_0^{rh} \frac{e^{-\frac{S}{\bar{S}}}}{\bar{S}} dS + \delta(h - \Delta h) \int_{rh}^{\infty} \frac{e^{-\frac{S}{\bar{S}}}}{\bar{S}} dS$$
$$= \delta(\Delta h)\left(1 - e^{-\frac{rh}{\bar{S}}}\right) + \delta(h - \Delta h) e^{-\frac{rh}{\bar{S}}}, \qquad [14]$$

where $\delta(.)$ is the Dirac delta distribution and we used the exponential distribution of the size S of HWEs, with mean $\bar{S}$.

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Similarly to the stochastic soil moisture dynamics at a point (4), Eq. **2** can be written as a Chapman–Kolmogorov forward equation for the evolution of the PDF of barrier elevation $f(h, t)$:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial h}[\rho(h)f(h,t)] - \lambda_0 f(h,t) + \lambda_0 \int_h^\infty f(h',t) f_E(h'-h, h')\mathrm{d}h'. \quad [15]$$

Substituting f_E (Eq. **14**) and integrating leads to Eq. **3**.

Definition of the Auxiliary Function $\phi(\xi)$. The function $\phi(\xi)$ is defined as

$$\phi(\xi) = e^{-\int_0^\xi \frac{\lambda^+(\xi')}{\rho^+(\xi')}\mathrm{d}\xi'} \quad [16]$$

and can be written in terms of the exponential integral $E_i(x) = -\int_{-x}^\infty x^{-1}e^{-x}\mathrm{d}x$ as

$$\phi(\xi) = \exp\left[-\lambda_0^+ e^{-\frac{1}{\xi_c}}\left(E_i\left(\frac{1}{\xi_c}\right) - E_i\left(\frac{1-\xi}{\xi_c}\right)\right)\right]. \quad [17]$$

Average Transport Rate due to Overwashes. Following energy considerations (2), the transport rate Q_S during a single overwash ($S > rh$) scales as $Q_S = (S-h)^2/T$, where T is a scale parameter with dimension of time. The expected value of Q_S per HWE is obtained by integrating over all possible overwashes:

$$\overline{Q}_S(t) = \int_0^H \frac{f(h/H, t)}{H}\left(\int_{rh}^\infty Q_S(S,h)\frac{e^{-\frac{S}{\bar{S}}}}{\bar{S}}\mathrm{d}S\right)\mathrm{d}h. \quad [18]$$

Substituting Q_S, introducing new variables $\eta = (S-h)/\bar{S}$ and $\xi = h/H$, and using $r = 1$ give

$$\overline{Q}_S(t) = \frac{\bar{S}^2}{T}\left(\int_0^\infty \eta^2 e^{-\eta}\mathrm{d}\eta\right)\int_0^1 e^{-\xi/\xi_c} f(\xi,t)\mathrm{d}\xi \quad [19]$$

$$= \frac{Q_0}{\lambda_0^+}\int_0^1 \lambda^+(\xi) f(\xi,t)\mathrm{d}\xi = Q_0 \frac{\overline{\lambda^+(t)}}{\lambda_0^+}, \quad [20]$$

where we used the definition $\lambda^+(\xi) = \lambda_0^+ e^{-\xi/\xi_c}$ and define $Q_0 = \frac{\bar{S}^2}{T}\left(\int_0^\infty \eta^2 e^{-\eta}\mathrm{d}\eta\right)$.

Data Availability. All study data are included in this article.

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